Variational Equation of a Ballistic Trajectory and Some of Its Applications

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The variational (perturbation, linearized) equation of ballistic motion is used to derive the state transition matrix and linear approximations to the solutions of typical two-point boundary-value guidance problems. Realistic gravitational and aerodynamic models, including tabular functions representing air density and the axial force coefficient of the re-entry vehicle, are used to obtain numerical results that demonstrate the accuracy of these linear approximations. In addition, this paper includes numerical results which show that aerodynamic phenomena completely dominate the solutions to the variational equation at low altitudes. Hence, a numerical comparison of the solutions of the variational equation with divided differencing and explicit techniques show that although divided differencing compares in accuracy with the variational equation to two or three digits the explicit method does not generally agree even in the first digit. Simulation results are provided which show that guided, perturbed boosters tend to thrust-terminate near a nominal ballistic trajectory (although not necessarily near the nominal thrust-termination point). Since the solutions of the variational equation can be updated along a nominal ballistic trajectory, the distance over which the linear approximations must be valid can be reduced considerably. Moreover, the variational equation can be solved in anywhere between one-half to one-fifth the time required to obtain divided differencing results. Thus, the variational equation is faster and more accurate than the other common methods of approximating the state transition matrix.

Introduction

ONE important problem in guidance is approximation of the solutions to boundary-value problems over suitable regions. Two methods of approximation are the so-called explicit and delta equations.[1] In the explicit method it is usually assumed that the dynamics of ballistic motion can be represented locally using only a spherical gravitational potential; when high accuracy is required and significant atmospheric drag effects exist, this method requires an accurate reference ballistic trajectory. The delta equations attempt to express, in a Taylor's series about a reference point, the solutions to the actual boundary-value problem as functions of the initial conditions. To approximate the linear Taylor's series coefficients, one may perturb slightly the appropriate initial condition of the reference trajectory, solve (usually numerically) the equations of motion, and approximate the corresponding first-order partial derivative of the solution with respect to the initial condition by divided differencing.[1] An improvement in accuracy and computing speed can be obtained by using instead the variational equation of the differential equations of motion. (These equations are frequently employed in space navigation[2] and optimal control problems.[3])

In this paper we pursue the approximation problem for realistic gravitational and aerodynamic models and illustrate with simulation results the accuracy of this technique in providing solutions to typical boundary-value problems arising in guidance.

Prior to describing the application of the variational equation, let us recall (from Ref. 4, except where noted) certain mathematical facts. If we have an ordinary vector differential equation

$$d\mathbf{y}/dt = \mathbf{f}(\mathbf{y},t) \tag{1}$$

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then, under a reasonable hypothesis for the vector function $\mathbf{f}$, for any point $(\mathbf{y}_o,t_o)$ there exists a unique solution $\mathbf{y}(t)$ to Eq. (1) satisfying $\mathbf{y}(t_o) = \mathbf{y}_o$. Hence, $\mathbf{y} = \mathbf{y}(\mathbf{y}_o,t_o,t)$. If the components of $\mathbf{f}$ have continuous, first-order partial derivatives with respect to the components of $\mathbf{y}$, then to each solution of Eq. (1) there is a corresponding linear, homogeneous vector differential equation

$$d\mathbf{z}/dt = (\partial\mathbf{f}/\partial\mathbf{y})\mathbf{z} \tag{2}$$

where the matrix of partial derivatives $\partial\mathbf{f}/\partial\mathbf{y}$ is assumed to be evaluated at time t using the reference solution $\mathbf{y}(\mathbf{y}_o,t_o,t)$. Equation (2) is called the variational equation of Eq. (1) (see Ref. 3 and pp. 55–57 of Ref. 5) (also referred to as a perturbation or linearized equation). Its solutions are linear approximations of how perturbations in $(\mathbf{y}_o,t_o)$ propagate as functions of time. To be more precise, Eq. (2) should be termed the first-variational equation of Eq. (1) since, under reasonable conditions, differential equations involving higher-order partial derivatives of $\mathbf{f}$ can be obtained (see p. 31 of Ref. 6).

Under the foregoing conditions, it is known that the components of $\mathbf{y}(\mathbf{y}_o,t_o,t)$ have first-order partial derivatives with respect to t_o and the components of $\mathbf{y}_o$, which are the solutions of Eq. (2) obtained by suitably varying the initial condition $\mathbf{z}(t_o)$. Indeed, if we denote the fundamental (state transition) matrix of Eq. (2) by $\Phi(t,t_o)$, where $\Phi(t_o,t_o)$ is the identity matrix and thus guarantees the uniqueness of $\Phi(t,t_o)$, then $\partial\mathbf{y}/\partial\mathbf{y}_o = \Phi(t,t_o)$ and

$$\partial\mathbf{y}/\partial t_o = -\Phi(t,t_o)\mathbf{f}(\mathbf{y}_o,t_o) \tag{3}$$

Thus, $\partial\mathbf{y}/\partial t_o$ can be obtained algebraically once the fundamental matrix is known. The fundamental matrix can be occasionally approximated without numerically integrating Eq. (2) (see pp. 75–76 of Ref. 6 and Ref. 7). These approximations have an important application in obtaining the homogeneous, as well as the particular, solutions to certain linear differential equations (dynamic equations of velocity-to-be-gained) associated with the problem of steering boosters.[8]

A most important aspect of the solutions of the variational equation is that for any time t_1

$$\Phi(t,t_1) = \Phi(t,t_o)\Phi(t_o,t_1) = \Phi(t,t_o)[\Phi(t_1,t_o)]^{-1} \quad (4)$$

and thus $\partial\mathbf{y}/\partial t_1 = -\Phi(t,t_1)\mathbf{f}[\mathbf{y}(\mathbf{y}_o,t_o,t_1),t_1]$. Equation (4) is termed an updating of the fundamental matrix. It plays a decisive role in the application of the variational equation to guidance problems, because the calculation of $\Phi(t,t_o)$ is usually time consuming [the long distance between an expected thrust-termination point and a target is usually large; i.e., $t - t_o$ is large, whereas $t_o - t_1$ usually can be taken small, thus enabling rapid calculation of Eq. (4)].

It is a useful fact (see pp. 46–47 on Ref. 5) that if $\mathbf{f}$ is not an explicit function of t, then

$$\partial\mathbf{y}/\partial t_o = -(d\mathbf{y}/dt) \quad (5)$$

Using Eq. (5) we can compare Eqs. (1) and (3) to determine the accuracy of the fundamental matrix calculation when numerical methods are employed. Even though $\mathbf{f}$ may be an explicit function of t, we know that $\partial^2\mathbf{y}/\partial t\partial\mathbf{y}_o$ exist and are continuous; therefore, it follows that

$$\frac{\partial^2\mathbf{y}}{\partial\mathbf{y}_o\partial t} = \frac{\partial^2\mathbf{y}}{\partial t\partial\mathbf{y}_o} = \frac{d(\partial\mathbf{y}/\partial\mathbf{y}_o)}{dt} = \frac{\partial\mathbf{f}}{\partial\mathbf{y}}\Phi(t,t_o)$$

providing $\partial^2\mathbf{f}/\partial\mathbf{y}^2$ exist and are continuous. Similar algebraic expressions for $\partial^2\mathbf{y}/\partial t_o\partial t$ and $\partial^2\mathbf{y}/\partial t^2$ in terms of first-order partial derivatives can be derived. But, if $\mathbf{f}$ is not an explicit function of time, we also have $\partial^2\mathbf{y}/\partial\mathbf{y}_o\partial t_o$ and $\partial^2\mathbf{y}/\partial t_o^2$ using Eq. (5).

Some Boundary-Value Problems

For point-mass ballistic motion

$$\ddot{\mathbf{x}} \equiv d\dot{\mathbf{x}}/dt = \mathbf{g}(\mathbf{x},t) + \mathbf{a}(\mathbf{x},\dot{\mathbf{x}},t) \quad (6)$$

where $\mathbf{x}$, $\dot{\mathbf{x}}$, $\mathbf{g}$, and $\mathbf{a}$ are the position, velocity, gravitational acceleration, and aerodynamic acceleration vectors, respectively. To evaluate $\mathbf{g}$ consider the oblate potential function in the inertial, right-handed, Cartesian coordinate system (x_1,x_2,x_3) which is Earth-centered with axes x_1 and x_2 in the equatorial plane and x_3 through the north pole

$$U = -\frac{GM}{A}\left\{\frac{A}{r} + J\left(\frac{A}{r}\right)^3\left[\frac{1}{3} - \left(\frac{x_3}{r}\right)^2\right] + \frac{D}{35}\left(\frac{A}{r}\right)^5\left[3 - 30\left(\frac{x_3}{r}\right)^2 + 35\left(\frac{x_3}{r}\right)^4\right]\right\}$$

where $r = |\mathbf{x}|$, the magnitude of $\mathbf{x} = (x_1,x_2,x_3)$; GM is the universal gravitational constant times the mass of the Earth (1.40766×10^{16}ft^3/sec^2); A is the semimajor axis of the reference ellipsoid of revolution (2.092601×10^7ft); and J and D are dimensionless constants in the spherical harmonic expansion (0.001638 and 0.0000107, respectively). Then the components of $\mathbf{g}$ are the negatives of the first partials of U. In the first-order partial derivatives of the components of $\mathbf{g}$ we neglect the term whose coefficient was D because it has a relatively small effect. (The numerical results reported later support this assumption.)

The aerodynamic acceleration is $\mathbf{a} = -(\rho/2\beta)|\dot{\mathbf{x}}_r|\dot{\mathbf{x}}_r$ where $\beta = m/C_AS$ (m is the mass, C_A the axial force coefficient, and S the aerodynamic reference area), and $\dot{\mathbf{x}}_r$ is the velocity vector with respect to the air. $\dot{\mathbf{x}}_r = \dot{\mathbf{x}} - \omega \times \mathbf{x}$ and the magnitude of ω, the earth's rotation vector about the polar axis x_3, is 7.29211×10^{-5} rad/sec. Altitude is given by $h = r - r_{SL}$ where

$$r_{SL} = A(1-e)/\{1 - (2e - e^2)[1 - (x_3/r)^2]\}^{1/2}$$

The ellipticity e of the reference ellipsoid is 1/297. Air density ρ and speed of sound v_s are tabular functions of altitude,[9] and the interpolating polynomials used to evaluate them were linear. Table 1 shows β as a function of Mach number M_a, which is given by $M_a = |\dot{\mathbf{x}}_r|/v_s$. (We assume that aerodynamic effects perpendicular to the roll axis are negligible.) The interpolating polynomial used to evaluate β also was linear. The appropriate partial derivatives of $\mathbf{a}$ involve partial derivatives of β, ρ, and v_s; and, after some experimentation, the procedure adopted to obtain these derivatives is to note that, if those functions are locally represented by interpolating polynomials, then their derivatives are locally represented by the derivatives of those polynomials. The validity of this result depends heavily upon the general smoothness of the data and upon using interpolation formulas whose order is not higher than linear or quadratic.

Table 1 Tabular values of β, slug/ft^2, vs Mach number

M_a	β	M_a	β	M_a	β
≤0.35	23.8	1.05	9.9	2.25	15.4
0.50	23.3	1.10	9.6	3.00	18.2
0.65	22.2	1.15	9.6	4.00	21.7
0.80	20.0	1.25	10.1	5.00	24.4
0.95	13.5	1.45	11.2	7.50	28.6
1.00	10.6	1.70	12.8	≥12.00	31.3

The fundamental matrix of the variational equation of Eq. (6) is

$$\Phi(t,t_o) = \begin{bmatrix} \partial\mathbf{x}(t)/\partial\mathbf{x}_o & \partial\mathbf{x}(t)/\partial\dot{\mathbf{x}}_o \\ \partial\dot{\mathbf{x}}(t)/\partial\mathbf{x}_o & \partial\dot{\mathbf{x}}(t)/\partial\dot{\mathbf{x}}_o \end{bmatrix}$$

Thus, to compute this fundamental matrix 42 scalar, ordinary differential equations must be simultaneously solved [Eq. (6) and its variational equation with the appropriate initial conditions]. Equation (3) can be used to obtain $\partial\mathbf{x}/\partial t_o$ and $\partial\dot{\mathbf{x}}/\partial t_o$.

The first boundary-value problem we will consider (the theoretical considerations in the remainder of this section do not depend upon the specific formulas for $\mathbf{g}$ and $\mathbf{a}$) is the constant total time-of-flight (t_T) constraint, which is given by the requirement that

$$\mathbf{x}_T(t_T) - \mathbf{x}(\mathbf{x}'_o,\dot{\mathbf{x}}'_o,t'_o,t_T) = 0 \quad (7)$$

be satisfied for initial conditions $(\mathbf{x}'_o,\dot{\mathbf{x}}'_o,t'_o)$, where $\mathbf{x}_T(t_T)$ denotes the target position vector at the desired time-of-flight t_T. Assume that Eq. (7) is satisfied for a specific ste of initial conditions $(\mathbf{x}_o,\dot{\mathbf{x}}_o,t_o)$, that the variational equation of Eq. (6) from t_o to t_T along the ballistic trajectory defined by $(\mathbf{x}_o,\dot{\mathbf{x}}_o,t_o)$ has been solved, and that the submatrix $\partial\mathbf{x}/\partial\dot{\mathbf{x}}_o$ of the fundamental matrix is invertible, which for all realistic trajectories we have found it to be. Then, by the Implicit Function Theorem, $\dot{\mathbf{x}}'_o$ is a function of $\mathbf{x}'_o$ and t'_o in a suitable neighborhood of $(\mathbf{x}_o,t_o)$ and is called the velocity-required vector $\mathbf{v}_r$ for the constant t_T constraint with the property that

$$\mathbf{x}_T(t_T) - \mathbf{x}[\mathbf{x}'_o,\mathbf{v}_r(\mathbf{x}'_o,t'_o),t'_o,t_T] = 0 \quad (8)$$

for $(\mathbf{x}'_o,t'_o)$ in this neighborhood. In this case, $\partial\mathbf{v}_r/\partial\mathbf{x}_o$ and $\partial\mathbf{v}_r/\partial t_o$ exist at $(\mathbf{x}_o,t_o)$ and are given by

$$\partial\mathbf{v}_r/\partial\mathbf{x}_o = -(\partial\mathbf{x}/\partial\dot{\mathbf{x}}_o)^{-1}(\partial\mathbf{x}/\partial\mathbf{x}_o) \quad (9)$$

and $\partial\mathbf{v}_r/\partial t_o = -(\partial\mathbf{x}/\partial\dot{\mathbf{x}}_o)^{-1}(\partial\mathbf{x}/\partial t_o)$. We see that in a suitable neighborhood of $(\mathbf{x}_o,t_o)$, $\mathbf{v}_r(\mathbf{x}'_o,t'_o)$ serves as the initial velocity vector at $(\mathbf{x}'_o,t'_o)$ of a ballistic trajectory, which satisfies the constant t_T constraint. Thus,

$$\dot{\mathbf{x}}'_o = \mathbf{v}_r(\mathbf{x}'_o,t'_o) \quad (10)$$

in this neighborhood, and the total derivative of Eq. (10) with respect to t'_o is $\ddot{\mathbf{x}}'_o = (\partial\mathbf{v}_r/\partial\mathbf{x}'_o)\dot{\mathbf{x}}'_o + \partial\mathbf{v}_r/\partial t'_o$, which can be rearranged to yield

$$\partial\mathbf{v}_r/\partial t_o = \mathbf{g}(\mathbf{x}_o,t_o) + \mathbf{a}[\mathbf{x}_o,\mathbf{v}_r(\mathbf{x}_o,t_o),t_o] - (\partial\mathbf{v}_r/\partial\mathbf{x}_o)\mathbf{v}_r(\mathbf{x}_o,t_o) \quad (11)$$

at $(\mathbf{x}_o,t_o)$ using Eqs. (6) and (10). Equation (11) can be used for any point $(\mathbf{x}'_o,t'_o)$ at which we know $\mathbf{v}_r$ and $\partial\mathbf{v}_r/\partial\mathbf{x}'_o$. Rather than updating the fundamental matrix, as in Eq. (4), and then using Eq. (9), it is possible to update $\partial\mathbf{v}_r/\partial\mathbf{x}_o$ directly to any point on the trajectory defined by $(\mathbf{x}_o,\dot{\mathbf{x}}_o,t_o)$ (even within the atmosphere) by solving a suitable matrix Riccati equation [see Eq. (6.57) of Ref. 2]; this requires the simultaneous solution of 15 scalar, ordinary differential equations, including Eq. (6), rather than the 42 required in the fundamental matrix update. However, the fundamental matrix contains more information than $\partial\mathbf{v}_r/\partial\mathbf{x}_0$. It is possible to derive differential equations that the second-order partial derivatives of $\mathbf{v}_r$ obey,[10] and it is possible to obtain $(\partial\mathbf{v}_r/\partial\mathbf{x}_o)^{-1}$ directly without first solving the variational equation of Eq. (6) [see Eq. (6.59) of Ref. 2].

The second boundary-value problem we will consider is the nonconstant total time-of-flight (t_T) constraint, which is given by the requirement that Eq. (7) be satisfied where t_T is not fixed. Assume that Eq. (7) is satisfied for a specific set of conditions $(\mathbf{x}_o,\dot{\mathbf{x}}_o,t_o,t_T)$ and that the fundamental matrix of the variational equation of Eq. (6) from t_o to t_T along the trajectory defined by $(\mathbf{x}_o,\dot{\mathbf{x}}_o t_o,t_T)$ has been evaluated. If the matrix whose columns are shown in partitioned form as

$$[-\partial\mathbf{x}_o/\partial\dot{x}_{1o}|-\partial\mathbf{x}_o/\partial\dot{x}_{2o}|\dot{\mathbf{x}}_T(t_T) - \dot{\mathbf{x}}(\mathbf{x}_o,\dot{\mathbf{x}}_o,t_o,t_T)]$$

is invertible, which for all realistic trajectories we have found it to be, then $\dot{x}'_{1o},\dot{x}'_{2o}$, and t'_T are functions of $\mathbf{x}'_o,t'_o$, and $\dot{x}'_{3o}$ in a suitable neighborhood of $(\mathbf{x}_o,t_o,\dot{x}_{3o})$, so that Eq. (7) is satisfied when a substitution similar to Eq. (8) is made. These two components of velocity (now denoted as v_{r1} and v_{r2}) are called the components of the velocity-required vector for the nonconstant t_T constraint. The first-order partial derivatives of v_{r1},v_{r2}, and t'_T at $(\mathbf{x}_o,t_o,\dot{x}_{3o})$ are given similar to Eq. (9). Also noting that $\dot{x}'_{1o} = v_{ri}(\mathbf{x}'_o, t'_o,\dot{x}'_{3o})$, $i = 1,2$, in this neighborhood, we proceed as in the transition from Eq. (10) to Eq. (11) to obtain

$$\partial v_{ri}/\partial t_o = g_i(\mathbf{x}_o,t_o) + a_i(\mathbf{x}_o,v_{r1},v_{r2},\dot{x}_{3o},t_o) - \sum_{j=1}^{2} [(\partial v_{ri}/\partial x_{jo})v_{rj}] - (\partial v_{ri}/\partial x_{3o})\dot{x}_{3o} - (\partial v_{ri}/\partial \dot{x}_{3o})[g_3(\mathbf{x}_o,t_o) + a_3(\mathbf{x}_o,v_{r1},v_{r2},\dot{x}_{3o},t_o)] \qquad i = 1,2$$

It is possible to derive a system of differential equations that the first-order partial derivatives of v_{r1} and v_{r2} with respect to the components of position and the third component of velocity obey along a reference ballistic trajectory.[11] Thus, these partial derivatives may also be updated directly.

Assume that the target is within the atmosphere and that we wish to constrain the re-entry angle γ to some reference value. This constraint may occur because some aerodynamic phenomena are closely correlated to γ, and it yields a three-component velocity-required vector. Other natural guidance constraints and objectives may arise (see pp. 199–201 of Ref. 1). We note that the steering law for the constant t_T constraint[8] also will work in those guidance problems where a three-component velocity-required vector exists.

Simulation Results

A single basic trajectory was used to obtain the results herein, but the results are representative of a greater variety of cases. Tables 2 and 3 were obtained by integrating the appropriate equations of motion and, in some cases, the associated variational equation by means of a Runge-Kutta fourth-order numerical integration routine along a reference trajectory that impacts the earth at about 5000-mile range with γ approximately 110° from the local vertical. The exoatmospheric and atmospheric integration step-sizes were 10 sec and 0.5 sec, respectively.

For sea level ($h = 0$), columns 1, 4, and 5 in Table 2 show the results of three methods of approximating some of the elements of the fundamental matrix of the variational equation. Column 1 holds solutions of the variational equation from t_o to t_T (the time of zero altitude on the reference trajectory). The results in column 4 were obtained by perturbing separately t_o and each component of initial position and velocity, then numerically integrating the differential equations of motion to t_T (which is not the time of zero altitude on the perturbed trajectory) and forming the divided-difference approximation to the appropriate first-order partial derivative. These initial perturbations were ±1 sec in time, ±1000 ft in position, and ±5 fps in velocity. Column 5 was obtained by the same approach as column 4 except that the differential equations were based upon only a spherical gravitational potential. The tra-

Table 2 Comparison of various methods of approximating some elements of the fundamental matrix and their variation during re-entry

	Method of approximation				
	Variational equations (actual dynamics)	Variational equations (actual dynamics)	Variational equations (actual dynamics)	Divided differencing (actual dynamics)	Divided differencing (spherical dynamics)
Altitude, ft	0	106,000	300,000	0	0
$\partial x_1/\partial x_{1o}$	−1.033	−0.198	−0.141	−1.027	−0.191
$\partial x_1/\partial x_{2o}$	−1.372	0.163	0.225	−1.363	0.198
$\partial x_1/\partial x_{3o}$	2.335	−0.171	−0.273	2.335	−0.225
$\partial x_2/\partial x_{1o}$	−1.047	0.267	0.316	−1.037	0.299
$\partial x_2/\partial x_{2o}$	−2.263	0.197	0.311	−2.248	0.244
$\partial x_2/\partial x_{3o}$	3.309	−0.684	−0.834	3.309	−0.777
$\partial x_3/\partial x_{1o}$	1.059	−1.304	−1.329	1.039	−1.394
$\partial x_3/\partial x_{2o}$	1.922	−2.479	−2.523	1.893	−2.648
$\partial x_3/\partial x_{3o}$	−2.938	4.242	4.317	−2.935	4.505
$\partial x_1/\partial \dot{x}_{1o}$	1205	1016	1017	1203	1005
$\partial x_1/\partial \dot{x}_{2o}$	151	68	66	150	66
$\partial x_1/\partial \dot{x}_{3o}$	2774	189	86	2774	460
$\partial x_2/\partial \dot{x}_{1o}$	414	103	93	412	96
$\partial x_2/\partial \dot{x}_{2o}$	1258	1125	1129	1256	1116
$\partial x_2/\partial \dot{x}_{3o}$	4167	43	−109	4168	470
$\partial x_3/\partial \dot{x}_{1o}$	−798	−237	−223	−793	−222
$\partial x_3/\partial \dot{x}_{2o}$	−874	−620	−604	−871	−619
$\partial x_3/\partial \dot{x}_{3o}$	−3684	3735	3817	−3683	3015
$\partial x_1/\partial t_o$	773	−5892	−6352	775	−5937
$\partial x_2/\partial t_o$	−583	−11153	−11833	−580	−11258
$\partial x_3/\partial t_o$	785	19400	−9299	786	20256

jectories are termed explicit, because they can be obtained from any one of several solutions to the classical two-body problem. Because at t_T these explicit trajectories yield $h \ll 0$, the time corresponding to $h = 0$ on the reference explicit trajectory was used in place of t_T. Note that, although the solutions of the variational equation and the divided differencing technique applied to the actual dynamics seem to agree fairly well (compare, however, parts I and II of Table 3), the results using explicit trajectories are remarkably different from the other two. This indicates, in part, the difficulty in applying explicit guidance techniques to problems where significant aerodynamic effects exist and high accuracy is required.

The data in columns 1–3 of Table 2 clearly show the large effect of aerodynamic acceleration upon some of the elements of the fundamental matrix. On the reference trajectory, $\mathbf{a}$ is reasonably small down to $h < 100{,}000$ ft, where it becomes rather large, thus significantly changing both the reference trajectory and the fundamental matrix thereafter. Less change would occur if γ or β were greater. Notice also that columns 3 and 5 in Table 2 compare favorably, further illustrating the lack of aerodynamic effects in explicit equations.

Taking $\mathbf{x}_T(t_T)$ as the position and time for $h = 0$ on the reference trajectory, we solved for $\partial \mathbf{v}_r/\partial \mathbf{x}_o$ and $\partial \mathbf{v}_r/\partial t_o$ for the constant t_T constraint as in Eqs. (9) and (11) using the data from column 1 of Table 2. Using these partial derivatives and selected values of the perturbations $\Delta t_o = t'_o - t_o$ and $\Delta \mathbf{x}_o = \mathbf{x}'_o - \mathbf{x}_o$, the linear approximation $\dot{\mathbf{x}}_o + \Delta \mathbf{v}_r$ of $\mathbf{v}_r$ were calculated, the differential equations of actual motion were integrated to time t_T with initial conditions $(\mathbf{x}'_o, \dot{\mathbf{x}}_o + \Delta \mathbf{v}_r, t'_o)$ in the same manner as the reference trajectory, and misses, $M \equiv |\mathbf{x}_T(t_T) - \mathbf{x}(\mathbf{x}'_o, \dot{\mathbf{x}}_o + \Delta \mathbf{v}_r, t'_o, t_T)|$, were calculated. Part I of Table 3 shows the results obtained from positive perturbations. (Negative perturbations gave practically identical misses, as one would expect, because miss is proportional to second-order terms.) The results in part II of Table 3 were obtained in exactly the same manner except $\partial \mathbf{v}_r/\partial \mathbf{x}_o$ and $\partial \mathbf{v}_r/\partial t_o$ were calculated, using the data in column 4 of Table 2. A comparison of parts I and II of Table 3 shows how important two or three extra digits of accuracy in some of the elements of the fundamental matrix can be.

With the same reference trajectory and $\mathbf{x}_T(t_T)$ used in the constant t_T constraint, the target was assumed to be Earth-fixed, and thus its position and velocity at any time t'_T were known. Hence, imposing the nonconstant t_T constraint, we solved for the first-order partial derivatives of v_{r1}, v_{r2}, and t'_T, using the data from column 1 of Table 2, which were then used to calculate the linear approximations $\dot{x}_{1o} + \Delta v_{r1}$, $\dot{x}_{2o} + \Delta v_{r2}$, and $t_T + \Delta t_T$ of v_{r1}, v_{r2}, and t'_T for selected values of Δt_o, $\Delta \mathbf{x}_o$, and $\Delta \dot{x}_{3o}$. The equations of motion were then integrated with the initial conditions $(\mathbf{x}'_o, \dot{x}_{1o} + \Delta v_{r1}, \dot{x}_{2o} + \Delta v_{r2}, \dot{x}'_{3o}, t'_o)$ to $t_T + \Delta t_T$, and misses, $M \equiv |\mathbf{x}_T(t_T + \Delta t_T) - \mathbf{x}(\mathbf{x}'_o, \dot{x}_{1o} + \Delta v_{r1}, \dot{x}_{2o} + \Delta v_{r2}, \dot{x}'_{3o}, t'_o, t_T + \Delta t_T)|$, were calculated (part III of Table 3). (Again negative perturbations produce practically no change in miss.) For a given perturbation, the miss for the nonconstant t_T constraint is about one-half of that for the constant t_T constraint for this reference trajectory.

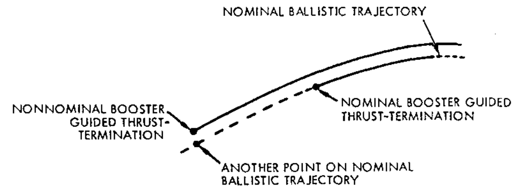

Fig. 1 Illustration of relation between non-nominal booster guided thrust-termination and nominal ballistic trajectory.

As noted previously, the fundamental matrix can be updated to any point on the reference trajectory; in fact, the first-order partial derivatives of the components of the appropriate velocity-required vector can be updated directly. The importance of this fact is that guided boosters, with non-nominal characteristics, tend to thrust-terminate much closer to a nominal ballistic trajectory than to the nominal booster guided thrust-termination point (Fig. 1). Actually, the initial conditions for the reference trajectory used to generate the results in Tables 2 and 3 were obtained by simulating the nonguided powered flight of a booster. With the components of $\mathbf{v}_r$ for the constant t_T constraint accurately approximated, using the method of discrete least-mean-squares by appropriate polynomials in a suitable region containing the nominal thrust-termination position and time, and with the appropriate coefficients for velocity steering in the atmosphere generated,[1] guided powered flights were simulated with both nominal (F_{nom}) and nonnominal ($1.06\, F_{\text{nom}}$) booster thrust characteristics using the exoatmospheric steering law in Ref. 8. The important results are recorded in Table 4. By integrating the equations of ballistic motion backward in time, using the nominal booster guided thrust termination position, velocity, and time as initial conditions, a point was found much closer to the nonnominal guided thrust-termination position and velocity. Thus the accuracy in using linear approximations can be significantly enhanced by suitably updating the fundamental matrix or $\partial \mathbf{v}_r/\partial \mathbf{x}_o$.

Computational Aspects and Concluding Remarks

Computational feasibility depends, of course, upon both computational capability available and the specific mission

Table 3 Accuracy of linear approximations to velocity-required for both constant and nonconstant t_T constraint

				Initial perturbations and resulting misses[a]					
Δt_o, sec	(M), ft	Δx_{1o}, kft	(M), ft	Δx_{2o}, kft	(M), ft	Δx_{3o}, kft	(M), ft	$\Delta \dot{x}_{3o}$, fps	(M), ft
I) Variational technique, constant t_T constraint									
1	(20)	10	(4)	10	(13)	10	(4)		
3	(50)	30	(70)	30	(61)	30	(85)		
5	(114)	50	(198)	50	(151)	50	(246)		
II) Divided differencing technique, constant t_T constraint									
1	(44)	10	(223)	10	(373)	10	(61)		
3	(144)	30	(615)	30	(1135)	30	(253)		
III) Variational technique, nonconstant t_T constraint									
2	(2)	30	(58)	30	(14)	30	(17)	5	(7)
4	(3)	60	(198)	60	(45)	60	(68)	10	(12)
6	(3)	90	(508)	90	(109)	90	(149)	20	(36)

[a] Only the nonzero initial perturbations, in thousands of feet, are shown, and the resulting miss in feet, due to the error in the linear approximation, is shown in parentheses next to the perturbation.

Table 4 Illustration of reduction of perturbations by updating along reference ballistic trajectory

Reference conditions	Components of booster time, position, and velocity perturbations at guided thrust-termination[a]						
	Δt_o, sec	Δx_{1o}, ft	Δx_{2o}, ft	Δx_{3o}, ft	$\Delta \dot{x}_{1o}$, fps	$\Delta \dot{x}_{2o}$, fps	$\Delta \dot{x}_{3o}$, fps
Nominal thrust-termination conditions[b]	−10	−24,587	−17,076	−85,325	−89	−146	100
Updated reference point[c]	−6	−4,228	7,983	418	−41	−64	23

[a] These perturbations were obtained by subtracting the reference state vector from the state vector at guided thrust-termination of the booster with a +6% thrust magnitude perturbation (1.06 F_{nom}).
[b] Booster time, position, and velocity, with nominal thrust characteristics (F_{nom}), at guided thrust-termination are t_o = 171 sec, x_{1o} = −8,350, 446 ft, x_{2o} = −14,532,169 ft, x_{3o} = 13,503,238 ft, $\dot{x}_{1o}$ = 5,113 fps, $\dot{x}_{2o}$ = 6,306 fps, $\dot{x}_{3o}$ = 21,398 fps.
[c] Time, position, and velocity 4 sec earlier on ballistic trajectory defined by nominal booster thrust-termination.

requirements. However, some general observations can be made. Most ground-based computers could solve Eq. (6) and its variational equation. Moreover, one can generally obtain $\partial \mathbf{x}/\partial \mathbf{x}_o$, etc., faster and more accurately by this approach, at a slight storage penalty, than by using the divided-differencing technique. For example, on a modern, general purpose digital computer, it is possible to arrange the computation of the variational equation of Eq. (6) so that about 200 additional instructions are needed. Using a Runge-Kutta fourth-order numerical integration routine, both the variational equation and Eq. (6) were integrated simultaneously to obtain the results in the first column of Table 2 in less than three times the time required to integrate the equations of ballistic motion by themselves or one-half to one-fifth of the time required by the divided-differencing technique, depending upon whether one or two-sided perturbations were used. If the basic dynamics were to become more complicated through use of more sophisticated aerodynamic and gravity models, the variational equation might be affected very little so that the computing time ratios would become even smaller than one-half or one-fifth.

For airborne applications with some preflight computations, updating either the fundamental matrix or $\partial \mathbf{v}_r/\partial \mathbf{x}_o$ particularly enhances accuracy and can be done using a simple integration method, since no aerodynamic partial derivatives probably will be involved and the update usually occurs over relatively short time periods. If the airborne computer does not possess sufficient capacity to update the partial derivatives, the concept can still be used to compute the coefficients in a polynomial expression[1] by the method of least-mean-square more efficiently than the usual divided differencing and iteration method.[12]

In conclusion, using realistic gravitational and aerodynamic models in the differential equations of ballistic motion of a re-entry vehicle, we have shown that integrating the variational equation itself is more accurate than use of either divided-differencing or explicit techniques for approximating the fundamental matrix. Of three methods for obtaining linear approximations to the solutions of typical boundary-value problems, the variational equation is particularly advantageous, since first, its solution may be updated along a nominal ballistic trajectory and second, guided boosters tend to thrust-terminate close to nominal ballistic trajectories.

With the variational equation, linear approximations to the solution of realistic boundary-value problems can be easily achieved, whereas it is not clear that two-body (explicit) equations will handle certain atmospheric constraints. The updating capability particularly enhances the region over which the linear approximation holds. The central unsolved problem (where only inflight updating is allowed) is that of obtaining better-than-linear solutions in a computationally efficient manner. Some results in this direction have been achieved.[13] If this problem has a reasonable solution, then the variational equation concept would serve as a computationally feasible and sufficiently accurate tool in all guidance problems.

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